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LETTER TO THE EDITOR

Homology and Wess–Zumino terms

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Abstract. Hopf's formula relating the second homology group to the fundamental group is used in a discussion of the nature and quantisation of the Wess-Zumino topological terms.

General homological criteria for the existence of Wess-Zumino topological terms (WZT) in the Lagrangian have been proposed by Alvarez (1985) and Braaten *et al* (1985) which are supposed to supersede the homotopic criteria of Witten (1983).

Stated briefly the prescription is as follows. Assume spacetime to be compactified to the d-sphere S^d and assume that the field φ takes values in a manifold M. Then a WZT is possible if the homological group $H_{d+1}(M; \mathbb{Z})$ contains at least one $\mathbb{Z}$, i.e. if the $(d+1)$th Betti number, b_{d+1}, is greater than zero. The actual expression for the WZT is the integral of a $(d+1)$-form over a rational $(d+1)$-chain, C, on M, whose boundary is, up to integer d-cycles, the image of S^d in M via φ. The quantisation of the coefficient of the integral (which is further pulled back to an integral over an extended $(d+1)$-dimensional spacetime) follows by requiring the ambiguity in C, a rational cycle in general, to have no effect when the term is exponentiated, just as in the theory of the magnetic monopole (Witten 1983, Dirac 1931).

Braaten *et al* (1985) further claim that when $H_d(M; \mathbb{Z})$ has torsion the quantisation is not integral but multiply integral and that, if $H_d(M; \mathbb{Z})$ has a free part, an extra b_d numbers are required to specify the WZT.

The modest aim of this letter is to point out that these claims are not generally correct.

In the following all homology groups will be integral ones and so the coefficient group $\mathbb{Z}$ will not be indicated.

The argument of Braaten *et al* (1985) involves expanding $\varphi(S^d)$ as a general d-cycle on M. However $\varphi(S^d)$ is clearly be definition a *spherical* d-cycle and the spherical homology classes, $\Sigma_d(M)$, form a subgroup of $H_d(M)$. In fact $\Sigma_d(M)$ is the image of $\pi_d(M)$ in $H_d(M)$ under the Hurewicz (natural) homomorphism (e.g. Maunder 1970, p 322). Hence, very simply, if $\pi_d(M)$ is trivial there are no further conditions on the WZT and C is ambiguous up to an integral cycle. (This is only a sufficient condition.)

$\pi_d(M)$ is trivial for the n-torus if d is greater than one so that in the example $d = 2$ and $M = S^1 \times S^1 \times S^1$, considered by Braaten *et al* (1985), there are no extra requirements. However for $d = 2$, $M = S^2 \times S^1$ we have $\pi_2 = \Sigma_2 = H_2 = \mathbb{Z}$ and in this case an extra constant *is* needed in their definition of the WZT.

Thus, if we can evaluate $\pi_d(M)$, we can find Σ_d and proceed as in Braaten *et al* (1985) but replacing H_d by Σ_d.

If $d=2$ we can avoid evaluating $\pi_2(M)$, if we know $\pi_1(M)$, by employing Hopf's famous formula

$$H_2(M)/\Sigma_2(M)=H_2(\pi_1) \tag{1}$$

expressed in homological terms (e.g. Hu 1959, p 201)). This, sometimes, tells us the difference between H_2 and π_2. If π_1 is Abelian, $\pi_1(M)$ can be replaced by $H_1(M)$ and (1) is then purely homological, which means it is computable. For simplicity assume that π_1 is Abelian; then it is easy to show using Künneth's formula that

$$H_2(\pi_1)=H_2(T)+b_1H_1(T)+\tfrac{1}{2}b_1(b_1-1)\mathbb{Z}$$

where T is the torsion part of $H_1(M)$. For any given T, Künneth's formula can again be applied to give $H_2(T)$ and $H_1(T)$. Thus for $T=\mathbb{Z}_p$

$$H_2(\pi_1)=\tfrac{1}{2}b_1(b_1-1)\mathbb{Z}+b_1\mathbb{Z}_p$$

and for $T=\mathbb{Z}_p+\mathbb{Z}_q$,

$$H_2(\pi_1)=\tfrac{1}{2}b_1(b_1-1)\mathbb{Z}+b_1(\mathbb{Z}_p+\mathbb{Z}_q)+\mathbb{Z}_{(p,q)}. \tag{2}$$

A general formula can be given (Hopf 1942) but is not needed here; (p,q) is the highest common factor of p and q.

If $H_2(\pi_1)$ is compared with $H_2(M)$ we see that an extra $b_2-\frac{1}{2}b_1(b_1-1)$ constants are required. In general the number will be $b_2(M)-b_2(\pi_1)$. (If π_1 is free non-Abelian $b_2(\pi_1)=0$.)

As an example with torsion consider $M=S^2\times S^2/\mathbb{Z}_2$ (non-orientable). Then $H_2(\pi_1)=\mathbb{Z}_2$ and, from general considerations or Künneth's formula, $H_2(M)=\mathbb{Z}+\mathbb{Z}_2$ so that $\Sigma_2(M)=\mathbb{Z}$. (This result also follows easily from $\pi_2(M)=2\mathbb{Z}$.) Thus the quantisation is a normal, integral one and not one in even integers. The same conclusion holds for $M=S^1\times S^3/\mathbb{Z}_p\times S^3/\mathbb{Z}_q$ for which $H_2(\pi_1)$ is given by (2) with $b_1=1$ and equals $H_2(M)$. (Again this also follows most quickly from $\pi_2(M)=0$! The advantage of formula (1) only appears when $\pi_2(M)$ is difficult to find.)

For arbitrary dimension d the ideal procedure would be to find π_d and then construct Σ_d using the Hurewicz homomorphism. If the non-trivial homotopy of M starts at dimension $d-1$ then Hurewicz's theorem says that $\pi_{d-1}=H_{d-1}(M)$, $=\pi$ say. The analogue of Hopf's formula is now

$$H_d(M)/\Sigma_d(M)=H_d(\pi,d-1) \tag{3}$$

where (π,n) is a space, for example an Eilenberg–MacLane space, whose only non-trivial homotopy group is $\pi_n=\pi$. Again this is a purely homological formula.

Another formula results if $\pi_i(M)$ is trivial for $1<i<d$. Then we have

$$H_d(M)/\Sigma_d(M)=H_d(\pi_1)$$

(e.g. Hu 1959, p 201). For example if $M=S^d\times T^t$, where T^t is the t-torus, $\pi_1(M)$ is free Abelian with t generators. Then $H_d(\pi_1)$ is free Abelian with $\binom{t}{d}$ generators, if $t\geqslant d$, so from (3) we see that $b_{2d}-\binom{t}{d}$ extra constants are required. Of course in this example $\pi_d(M)=\mathbb{Z}$ so that Σ_d can be constructed directly. Other examples can be found in Hopf (1943).

For $d=2$ the necessary and sufficient condition that the statements of Braaten *et al* (1985) should be correct is that $H_2(\pi_1)$ be trivial. This will be the case if π_1 itself is trivial of course, but will also be true when π_1 consists of a single cyclic group or when it is free non-Abelian as, for example, when M is a 2-sphere minus three points.

Incidentally if π_1 is only known by its presentation as a group with n generators and r relations it is still possible to give an upper bound to the dimension of $H_2(\pi_1)$, i.e. to $b_2(\pi_1)$ (e.g. Rotman 1979). This yields a lower bound to the number of free generators in Σ_2 equal to $b_2 - r - b_1 + n$. (If π_1 is free non-Abelian $r = 0$ and $b_1 = n$.) It is thus seen that the statements of Braaten *et al* (1985) regarding the number of extra constants will be correct if $b_1 = n - r$.

In higher dimensions the necessary and sufficient conditions for $\Sigma_d(M)$ to equal $H_d(M)$ do not seem to have been worked out.

Turning to more general considerations it is known in the case of quantum mechanics on $R^3 - \{0\}$ that the existence of a free part to $H^2(M)$ corresponds to the possibility of there being a monopole on $\{0\}$. The curvature of the wavefunction (complex) line bundle is the magnetic field and has integral periods corresponding to magnetic charge quantisation (e.g. Kostant 1970, Simms and Woodhouse 1976, Greub and Petry 1975). $H^2(M)$ catalogues the isomorphism classes of complex line bundles over M (e.g. Cartan 1950) and a form realisation of the cohomology, which exhibits the Dirac string, can be found in Allendoerfer and Eells (1957) (see also Isham 1978).

In field theory the classes of wavefunctionals, $\Psi[\varphi]$, will similarly be catalogued by $H^2(M^X)$ where X is the spatial section of spacetime. We can take X to be compactified to S^{d-1} for topological purposes. It is our contention that the free part of this H^2 corresponds to the existence of 'functional monopoles'.

The relevant theory is that of the cohomology of iterated loop spaces.

If $d = 2$ a standard result can be applied if M is a sphere (e.g. Hu 1959, Bott and Tu 1982, p 203). It is known that

$$H^*(\Omega S^2) = \mathbb{Z} \qquad \text{in all dimensions}$$

and

$$\begin{aligned} H^*(\Omega S^n) &= Z \qquad \text{in dimensions } 0, n-1, 2(n-1), \ldots \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

For example $H^2(\Omega S^2) = \mathbb{Z}$ and $H^2(\Omega S^3) = \mathbb{Z}$, where $\Omega^n M$ is the iterated loop space M^{S^n}. In both cases functional monopole configuration exist, although there would be no WZT term for S^2, according to the rules of Alvarez. A discussion of the significance and analytical details of these monopoles is left for another time.

There is a subsidiary classification of quantum theories according to the character group of the fundamental group of the configuration space. It can bè expressed homologically as follows. The subsidiary classes are in one-to-one correspondence with $\mathrm{Hom}(\pi_1, \mathcal{R})$ where $\mathcal{R}$ is the group of real numbers modulo unity. (Equivalent to $\mathcal{R}$ are S^1, $U(1)$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$.) Now $\mathrm{Hom}(\pi_1, \mathcal{R}) = \mathrm{Hom}(H_1, \mathcal{R}) = b_1\mathcal{R} + T_1$ where T_1 is the torsion part of H_1. The classes are therefore labelled by b_1 real numbers, $\underset{\sim}{\alpha}$, each between 0 and $\frac{1}{2}$, and also by the torsion labels. The latter are already accounted for in the H^2 classification because H^2 and H_1 have the same torsion. If a non-zero b_2 corresponds to a monopole, a non-zero b_1 corresponds to an Aharonov–Bohm flux tube (Dowker 1979).

For quantum field theory π_1 is $\pi_d(M)$ which therefore equals $H_1(\Omega^{d-1}M)$. If M is simply connected we know that $\pi_2(M) = H_2(M)$ and so, in two dimensions, $H_1(\Omega M) = H_2(M)$. Since the free parts of the corresponding cohomology groups can be identified, the 1-form integrals on the functional space needed to remove the multivaluedness of the wavefunctional (Dowker 1979, 1985) can be replaced by 2-form integrals on the target space M. If one wishes to express the integral cohomology ('functional Chern') classes of $H^2(\Omega M)$ in terms of 3-forms on M, like the WZT, it is sufficient to have $\pi_1(M)$ and $\pi_2(M)$ trivial because then $H_2(\Omega M) = \pi_2(\Omega M) \equiv \pi_3(M) = H_3(M)$. $M = \mathrm{SU}(n)$ is a good example. In d dimensions the corresponding statement is that $\pi_i(M)$ should be trivial for $1 \leq i \leq d$. This is true for the sphere S^{d+1} but not for compact simple Lie groups, for which π_3 is $\mathbb{Z}$. However the condition is only a sufficient one. The necessary conditions remain to be worked out for $b_2(\Omega^{d-1}M)$ to equal $b_{d+1}(M)$.

That the above conditions are too stringent is seen by considering the standard case of $M = \mathrm{SU}(3)$ and $d = 4$, since then π_4 is trivial and $\pi_5 = H_5 = \mathbb{Z}$.

If π_d is trivial the classification of functional monopoles by the free part of $H^2(\Omega^{d-1}M)$, = free part of $H_2(\Omega^{d-1}M)$, is the same as Witten's homotopic classification since then $H_2(\Omega^{d-1}M) = \pi_{d+1}$. Incidentally, if $d = 7$ and $M = \mathrm{SU}(3)$ we have $\pi_7(M) = H_7(M) = 0$ and $H_8(M) = \mathbb{Z}$, $\pi_8 = \mathbb{Z}_{12}$. Thus, since π_8 has no free part, we would not expect a WZT according to Witten but because $H_8 = \mathbb{Z}$ there would be such a term according to Alvarez. The rational functional Chern classes are trivial in this case.

References

Allendoerfer C B and Eells J 1957 *Comm. Math. Helv.* **32** 165
Alvarez O 1985 *Commun. Math. Phys.* **100** 279
Bott R and Tu L W 1982 *Differential forms in algebraic topology* (Berlin: Springer)
Braaten E, Curtright T L and Zachos C K 1985 *Nucl. Phys.* B **260** 630
Cartan H 1950 *Seminaire Bourbaki* **34** 1
Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
Dowker J S 1979 *Selected topics in topology and quantum field theory* (Austin: University of Texas)
—— 1985 *J. Phys. A: Math. Gen.* **18** 3521
Greub W and Petry H 1975 *J. Math. Phys.* **16** 1347
Konstant B 1970 *Lecture Notes in Mathematics* **170** 87
Hopf H 1942 *Comm. Math. Helv.* **14** 257
—— 1943 *Comm. Math. Helv.* **15** 27
Hu S-T 1959 *Homotopy theory* (New York: Academic)
Isham C J 1978 *Proc. R. Soc.* A **362** 383
Maunder C R F 1970 *Algebraic Topology* (London: Van Nostrand)
Rotman J J 1979 *Introduction to homological algebra* (New York: Academic)
Simms D J and Woodhouse N M J 1976 *Lecture Notes in Physics* **53**
Witten E 1983 *Nucl. Phys.* B **223** 422